MATHEMATICS

LOCAL CONNECTEDNESS AND OTHER PROPERTIES OF GA COMPACTIFICATIONS

BY

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1. *Introduction*

In [2] DE GROOT and AARTS defined compactifications of T_1-spaces (we shall refer to them as GA compactifications) using certain kinds of subbases for the closed sets of the space. The GA compactifications are generalizations of Wallman compactifications studied in numerous papers (c.f. [5], [8], [9], [10]).

The results of section 2 are a continuation of the results of [2] in the sense that they are general results on GA compactifications. However, these results are motivated by the results of section 3, where the subbase has the added restriction that it consists of connected sets. Spaces satisfying this added restriction have been studied by DE GROOT in [4] and our results are a continuation of his.

Section 4 contains all of the proofs of the results in sections 2 and 3.

Section 5 contains a number of examples which show that our conditions are not necessary.

2. *General Theorems*

We first of all recall the notations of [2] that are needed in this paper and refer the reader to [2] for definitions that are used but not stated here.

Let X be a T_1-space, let $\mathscr{S}$ be a subbase for the closed sets of X which is subbase regular and subbase normal (i.e. from [2], X is a completely regular space). We shall sometimes refer to such a subbase as a "GA subbase". Let X' be the set of all maximal centered systems of X and for each $S \in \mathscr{S}$, let $S' = \{\xi' \in X' | S \in \xi'\}$. If $\mathscr{S}' = \{S' | S \in \mathscr{S}\}$, then $\mathscr{S}'$ is a subbase (for the closed sets) of X' and X' is a T_1-compactification of X. For each $\xi' \in X'$, let $\xi^* = \{S \in \mathscr{S} | S$ meets every member of $\xi'\}$. It follows that each ξ^* is a maximal linked system of $\mathscr{S}$ and the relation $\sim$ defined on X' by $\xi' \sim \eta'$ iff $\xi^* = \eta^*$ is an equivalence relation. The quotient space X^* of X' (with quotient map π) is a Hausdorff compactification of X. X^* is the GA compactification of X (with respect to $\mathscr{S}$). For each $S \in \mathscr{S}$, let $S^- = cl_{X^*} S$ and $S^* = \{\xi^* | S \in \xi^*\}$; let $\mathscr{S}^- = \{S^- | S \in \mathscr{S}\}$ and $\mathscr{S}^* = \{S^* | S \in \mathscr{S}\}$.

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are two collections of subsets of X, we say that $\mathscr{C}_1$ *screens*

$\mathscr{C}_2$ provided that for every two disjoint members A and B of $\mathscr{C}_2$, there is a finite cover of X by members of $\mathscr{C}_1$ such that no member of the cover intersects both A and B. If $\mathscr{S}$ is a subbase for X, we let $\mathscr{B}$ be the set of all finite unions of members of $\mathscr{S}$. $\mathscr{S}$ is said to be *basic* in case $\mathscr{S}$ screens $\mathscr{B} \cup \{\{x\}|x \in X\}$. Note that if $\mathscr{S}$ is basic, then $\mathscr{S}$ is subbase regular and subbase normal and $\mathscr{B}$ is (sub)base regular and (sub)base normal.

Theorem 1. *If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two subbase regular and subbase normal subbases for X such that $\mathscr{S}_1$ screens $\mathscr{S}_1 \cup \mathscr{S}_2$, then the GA compactification, $X_2{}^*$, of X with respect to $\mathscr{S}_2$ is smaller than or equal to the GA compactification, $X_1{}^*$, of X with respect to $\mathscr{S}_1$, in the usual ordering of Hausdorff compactifications of X.*

Remark 1. It follows immediately from this theorem that if $\mathscr{S}$ is basic, then $\mathscr{S}$ and $\mathscr{B}$ generate the same X^*. Thus, "basic" is a separation property of $\mathscr{S}$. If $\mathscr{S}$ is basic and consists of closed connected sets, it follows from theorems of [4] and [6] that X is locally connected.

For another comparison theorem in a more general setting see [3].

We assume that $\mathscr{S}$ is a subbase regular and subbase normal subbase for X.

Theorem 2. *$\mathscr{S}^-$ is a subbase normal subbase for X^*.*

Remark 2. The natural subbase for X^* is $\mathscr{S}^*$ (see [2]). In general $\mathscr{S}^-$ is not subbase regular. On the other hand, as was pointed out in [2], each $S^* \in \mathscr{S}^*$ need not necessarily be the closure of S in X^*.

Theorem 3. *For each $S \in \mathscr{S}$, $S^- = \pi(S')$.*

Theorem 4. *$\mathscr{S}^-$ is subbase regular iff $S^* = S^-(=\pi(S'))$ for all $S \in \mathscr{S}$.*

Given a finite intersection $I = \bigcap_1^n T_i$ such that $T_i \in \mathscr{S}$, $i = 1, \ldots, n$, and $S \in \mathscr{S}$ such that $S \cap I = \emptyset$, let property ${}_IP_S$ be: there exist choice functions f and g on finite covers (by members of $\mathscr{S}$) of S and I, respectively, such that $f(\mathscr{C}_S) \in \mathscr{C}_S$, $g(\mathscr{C}_I) \in \mathscr{C}_I$, and $f(\mathscr{C}_S) \cap g(\mathscr{C}_I) \neq \emptyset$ for any finite covers $\mathscr{C}_S$ of S and $\mathscr{C}_I$ of I.

Theorem 5. *π is a homeomorphism iff ${}_IP_S$ does not hold for any S and I with $S \cap I = \emptyset$.*

Theorem 6. *The following conditions on $\mathscr{S}$ are equivalent.*

1) *Members of $\mathscr{S}$ are screened by $\mathscr{S}$ from finite intersections of members of $\mathscr{S}$.*
2) *$\mathscr{S}'$ is subbase regular.*
3) *π is a homeomorphism and $\mathscr{S}^-$ is subbase regular.*

3. *Subbases of closed connected sets*

In [4], DE GROOT calls a space "connectedly generated (cg)" if it has a subbase for the closed sets consisting of connected sets. It is easy to see that these are just the semi-locally connected spaces of WHYBURN [11] and [12]. For *cg* spaces, the screening properties of the subbase of all closed connected sets are closely related to local connectedness; see [4] and [6]. In particular, the subbase $\mathscr{S}$ of all closeed connected sets is subbase normal (and consequently subbase regular) iff X is locally connected and every pair of disjoint closed connected sets are contained in disjoint open sets [6]. The local connectedness of X' and X^*, under these conditions, is the subject of another paper [HURSCH, 7].

Here, we give some general sufficient conditions for X^* to be locally connected if $\mathscr{S}$ consists of closed connected sets. It will be difficult to find necessary and sufficient general conditions. In fact, even when $\mathscr{S}$ is the subbase of all of the closed connected sets, no satisfactory necessary and sufficient condition is known for X^* to be locally connected.

If $\mathscr{S}$ consists of closed connected sets (as assumed in this section), then each $S' \in \mathscr{S}'$ is connected because the closure of a connected set is connected. Furthermore, since the continuous image of a connected set is connected, each $S^- \in \mathscr{S}^-$ is connected. Thus, we have the following corollary to theorem 2.

Corollary. *If $\mathscr{S}$ consists of closed connected sets, then X^* is c.g.*

Let $\mathscr{C}$ consist of all connected finite unions of members of $\mathscr{S}$.

Theorem 7. *If $\mathscr{C}$ is subbase regular and subbase normal and $\mathscr{C}^-$ is subbase regular, then X^* is locally connected.*

Corollary 1. *If $\mathscr{S}$ is basic and $\mathscr{B}$ screens finite intersections of members of $\mathscr{B}$ from members of $\mathscr{B}$, then X^* is locally connected.*

Corollary 2. *If $\mathscr{S}$ contains all of the points of X and screens members of $\mathscr{C}$ from finite intersections of members of $\mathscr{S}$, then X^* is locally connected.*

4. *Proofs*

In order to prove theorem 1, we define a relation R with domain X_1' and range X_2^* by the following: $\xi' R \eta^*$ iff there exists a maximal centered system ξ'' in $\mathscr{S}_1 \cup \mathscr{S}_2$ such that $\xi' \subset \xi''$ and $\xi'' \cap \mathscr{S}_2 \subset \eta^*$. The proof of theorem 1 follows immediately from lemma 5 below.

Lemma 1. *R is a function.*

Proof. Suppose that $\xi' R \eta_1^*$ and $\xi' R \eta_2^*$ for $\eta_1^* \neq \eta_2^*$ in X_2^*. There exist maximal centered systems ξ_1'' and ξ_2'' in $\mathscr{S}_1 \cup \mathscr{S}_2$ such that $\xi' \subset \xi_i''$ and $\xi_i'' \cap \mathscr{S}_2 \subset \eta_i^*$ for $i=1, 2$. Since $\eta_1^* \neq \eta_2^*$, there exist $T_i \in \eta_i^*$ such that $T_1 \cap T_2 = \emptyset$. Screen T_1 and T_2 by $\mathscr{S}_2$ to get an S_1 in ξ_1'' with $S_1 \cap T_2 = \emptyset$. Screen S_1 and T_2 by $\mathscr{S}_2$ to get an S_2 in ξ_2'' with $S_1 \cap S_2 = \emptyset$. By screening S_1 and S_2 by $\mathscr{S}_1$ to get one member of ξ' disjoint from S_2 and then repeating the process to get two disjoint members of ξ'; a contradiction.

Lemma 2. *If π_1 is the quotient mapping of X_1' onto X_1^*, then R induces a function f of X_1^* into X_2^* by $f(\pi_1(\xi')) = \eta^*$ iff $\xi' R \eta^*$.*

Proof. Use the method of the proof of lemma 1.

Lemma 3. *The range of f is all of X_2^*.*

Proof. For $\eta^* \in X_2^*$, let $\eta' \in X_2'$ such that $\eta' \subset \eta^*$. Let ξ'' be a maximal centered system of $\mathscr{S}_1 \cup \mathscr{S}_2$ such that $\eta' \subset \xi''$. Let ξ' be a maximal centered system of X_1' such that $\xi'' \cap \mathscr{S}_1 \subset \xi'$. Let ξ''' be a maximal centered system of $\mathscr{S}_1 \cup \mathscr{S}_2$ such that $\xi' \subset \xi'''$. If $\xi''' \cap \mathscr{S}_2 \subset \eta^*$ then we will have shown $\xi' R \eta^*$ and hence $f(\pi_i(\xi')) = \eta^*$. Suppose there is $S \in \xi''' \cap \mathscr{S}_2$ and $T \in \eta'$ such that $S \cap T = \emptyset$. Screen S and T by members of $\mathscr{S}_1$ to get $S_1 \in \xi'$ with $S_1 \cap T = \emptyset$. Screen S_1 and T by members of $\mathscr{S}_1$ to find $S_2 \in \xi''$ such that $S_2 \cap T = \emptyset$. This contradicts the fact that S_2, T are in ξ''. Thus $\xi''' \cap \mathscr{S}_2 \subset \eta^*$.

Lemma 4. *f keeps X fixed.*

Proof. Use the fact that $\mathscr{S}_1$ and $\mathscr{S}_2$ are both subbase regular.

Lemma 5. *f is continuous.*

Proof. If $S \in \mathscr{S}_2$, $\xi^* \in X_1^*$ and $f(\xi^*) \notin S^*$, let $T \in f(\xi^*)$ such that $T \cap S = \emptyset$. Screen T and S by members of $\mathscr{S}_1$ and let the members of the screen that meet T be $V_1, \ldots, V_r$ and the members that don't meet T be $V_{r+1}, \ldots, V_s$. For $i = r+1, \ldots, s$, screen V_i and T by members of $\mathscr{S}_1$ and let the members of the screen that don't meet T be $W_1^i, \ldots, W_{n_i}^i$. It can be shown that $\xi^* \notin (W_j^i)^*$ for any i, j and if $\psi^* \notin (W_j^i)^*$ for all i, j, then $f(\psi^*) \notin S^*$. Consequently, $f^{-1}(S^*)$ is closed and f is continuous.

Proof of theorem 2. To see that $\mathscr{S}^-$ is a subbase of X^*, let $S \in \mathscr{S}$ and $\xi^* \notin S^*$. Screen S^* and ξ^* by members of $\mathscr{S}^*$, and let $T_1^*, \ldots, T_n^*$ be those members of the screen which don't contain ξ^*. Since the members of the screen cover X^*, their traces on X are members of $\mathscr{S}$ which cover X. Thus $S^* \subset \bigcup_1^n T_i^-$ and $\xi^* \notin \bigcup_1^n T_i^-$. It easily follows that $\mathscr{S}^-$ is a subbase for X^*. $\mathscr{S}^-$ is trivially subbase normal.

Proof of theorem 3. Since π is a closed map, $S^- \subset \pi(S')$. On the other hand, if $\xi^* \in \pi(S')$, there exists $\eta' \in S'$ such that $\eta^* = \xi^*$. It follows that, if $T_1, \ldots, T_n$ are members of $\mathscr{S}$ which cover S then some $T_{i_0} \in \eta'$ and hence $\xi^* \in T_{i_0}^*$ and thus $\xi^* \in S^-$.

Proof of theorem 4. Suppose that $\mathscr{S}^-$ is subbase regular and $\xi^* \in S^* - S^-$. Screen ξ^* and S^- by members of $\mathscr{S}^-$ and let T^- be a number of the screen such that $\xi^* \in T^-$. By theorem 3, $T \in \xi^*$ and $T \cap S = \emptyset$, so $\xi^* \notin S^*$.

Conversely, if $S^* = S^-$ for all $S \in \mathscr{S}$, then $\mathscr{S}^- = \mathscr{S}^*$ and $\mathscr{S}^*$ is subbase regular.

Proof of theorem 5. Suppose that π is not a homeomorphism. Then there exist $\xi' \neq \eta'$ such that $\xi^* = \eta^*$. Since $\xi' \neq \eta'$, we can find

$S \in \xi' - \eta'$ and a finite intersection $I = \bigcap_1^n T_i$ of elements of η' such that $S \cap I = \emptyset$. For any cover $\mathscr{C}_S$ of S, let $f(\mathscr{C}_S)$ be a member of the cover which is in ξ' and for any cover $\mathscr{C}_I$ of I, let $g(\mathscr{C}_I)$ be a member of $\mathscr{C}_I$ which is in η'. Clearly ${}_IP_S$ holds.

If ${}_IP_S$ holds for $I = \bigcap_1^n T_i$, let ξ' be a centered system containing $T_1, \ldots, T_n$ and maximal with respect to the property that every cover of every finite intersection in ξ' has a member which meets $f(\mathscr{C}_S)$ for every cover $\mathscr{C}_S$ of S. Let $F_1, \ldots, F_m$ be a cover of X by members of $\mathscr{S}$. Suppose that for each i we can find a finite intersection I_i of members of ξ' such that $F_i \cap I_i$ has a cover $\mathscr{P}_i$, no member of which meets every $f(\mathscr{C}_S)$ for $\mathscr{C}_S$ a cover of S. Let $\mathscr{P} = \bigcup_1^m \mathscr{P}_i$ and let $J = \bigcap_1^m I_i$. Then $\mathscr{P}$ is a cover of J failing the assumption on ξ'. Thus ξ' contains a member of every finite cover of X. To see that ξ' is a maximal centered system of $\mathscr{S}$, let $R \in \mathscr{S}$ and suppose that there exists a finite intersection J of elements of ξ' and a cover $\mathscr{P}$ of $R \cap J$ such that no member of $\mathscr{P}$ meets $f(\mathscr{C}_S)$ for every cover $\mathscr{C}_S$ of S. Screening each member of $\mathscr{P}$ from some $f(\mathscr{C}_S)$, we obtain a finite intersection in ξ' which does not meet $R \cap J$. Thus R fails to meet some finite intersection in ξ'. We have then that ξ' is a maximal centered system of $\mathscr{S}$ and every member of ξ' meets every $f(\mathscr{C}_S)$.

Now let η' be a centered system containing S such that every cover of every finite intersection in η', by members of $\mathscr{S}$, has a member which meets every member of ξ'. Using a procedure similar to that for ξ', we can show that η' is a maximal centered system and every member of η' meets every member of ξ'. Thus $\pi(\eta') = \pi(\xi')$ and so π is not a homeomorphism.

Proof of theorem 6. Let $S \in \mathscr{S}$ and suppose that $\xi' \notin S'$. Then there exist $T_1, \ldots, T_n \in \xi'$ such that if $I = \bigcap_1^n T_i$, then $S \cap I = \emptyset$. Assuming 1) we screen S and I by $R_1, \ldots, R_m$. Then $R_1', \ldots, R_m'$ is a screen of S' and ξ'. Thus $\mathscr{S}'$ is subbase regular. Assuming 2) we can screen S' and ξ' by $R_1', \ldots, R_m'$. Some R_i not meeting S must be in ξ' and so $\xi' = \xi^*$; thus π is a homeomorphism. Furthermore, $\mathscr{S}^- = \pi(\mathscr{S}')$ is subbase regular. Assuming 3), suppose that S and $I = \bigcap_1^n T_i$ are disjoint and cannot be screened. Let ξ' be a centered system containing $T_1, \ldots, T_n$ and maximal with respect to the property that every cover of a finite intersection in ξ' has a member which meets S. As in the proof of theorem 5, ξ' can be shown to be a maximal centered system every member of which meets S. Clearly $\xi^* \in S^*$. If $\xi^* \notin \pi(S')$, then by theorem 4, $\mathscr{S}^-$ is not subbase regular. If $\xi^* \in \pi(S')$ then there is an $\eta' \in S'$ such that $\pi(\eta') = \pi(\xi')$ and π is not a homeomorphism.

Proof of corollary to theorem 2. $\mathscr{S}^-$ is a subbase and every S^- in $\mathscr{S}^-$ is the closure of a connected set and thus is connected.

Proof of theorem 7. We will prove that the collection of all closed connected subsets of X^* is subbase regular. If C is a closed connected subset of X^* and $\xi^* \notin C$, we may find $S_1, \ldots, S_n \in \mathscr{S}$ such that $C \subset \bigcup_1^n S_i^-$, $\xi^* \notin S_i^-$ for $i = 1, \ldots, n$, and $\bigcup_1^n S_i$ is connected. Since $\mathscr{C}$ is subbase regular and subbase normal, theorem 1 implies that $\mathscr{C}$ and $\mathscr{S}$ generate the same X^*. Since $\mathscr{C}^-$ is subbase regular and $\xi^* \notin \bigcup_1^n S_i^-$, we can screen ξ^* from $\bigcup_1^n S_i^-$ by members of $\mathscr{S}^-$. Each member of the screen is a closed connected set and the screen clearly screens C and ξ^*. Thus X^* is locally connected.

Proof of Corollary 1. By theorem 6, $\mathscr{B}^-$ is subbase regular. Thus $\mathscr{C}^-$ is and the rest follows from remark 1.

Proof of Corollary 2. By theorem 6, the conditions of theorem 7 are satisfied.

5. *Examples*

Example 1. This is just example 4 of [2]. X is a locally compact, non compact Hausdorff space. Let p and q be two points of X. Let $\mathscr{S}$ consist of

1. all compact subsets of X.
2. all closed subsets of X which contain at least one of the points p and q and the complements of which have a compact closure in X.

It is pointed out in [2], that X^* is the one point compactification of X, and if $S = \{p, q\}$ then

$$\{p, q, \infty\} = S^* \neq S^- = \{p, q\}.$$

We add that, since the noncompact sets of $\mathscr{S}$ are all in the unique free centered system of X', π is one to one.

Example 2. We consider the following variation of example 5.2. of [1]. Let X be the subset of the plane consisting of the segment joining $(-1, 1)$ to $(1, 1)$, the segment joining $(-1, -1)$ to $(1, -1)$, the segment joining $(-1, -1)$ to $(-1, 1)$, and the segments $\{(1/n, y)\colon -1 < y < 1\}$, $n = 1, 2, \ldots$. Then X is connected and locally connected but has no locally connected compactification. Let $\mathscr{S}$ consist of all those closed connected subsets of X which are the trace, on X, of rectangles in the plane, two of whose sides are parallel to the x-axis.

If S is the trace on X of $\{(x, y)\colon -1 \leqslant x \leqslant 0, -1 \leqslant y \leqslant 1\}$. Then, since S is compact, $S = S^-$ and $S^* = (X^* \setminus X) \cup S$. It is not hard to see that π is a homeomorphism. Furthermore $\mathscr{S}$ is basic. S is not screened from finite intersections of members of $\mathscr{S}$, and $\mathscr{S}'$ and $\mathscr{S}^-$ are not subbase regular.

Example 3. Let X be that subset of the plane consisting of the x-axis and the points $p=(-1, 1)$, $q=(0, 1)$, $r=(1, 1)$. Let a set be in $\mathscr{S}$ if

1. It is a compact connected subset of the x-axis.
2. $S=\{p\}$, or $S=\{q\}$, or $S=\{r\}$.
3. S consists of a noncompact closed connected subset of the x-axis plus at least two out of the three points p, q and r.

Then it is easy to see that

1. X' is the two point compactification of the x-axis plus the three isolated points p, q and r.
2. X^* is the one-point compactification of the x-axis plus p, q and r. So π is not $1-1$.
3. For all $S \in \mathscr{S}$, $S^-=S^*$ so $\mathscr{S}^-$ is subbase regular.
4. Both X' and X^* are locally connected and Hausdorff.

Example 4. This is just example 3 but we add $S=\{p, q\}$ to $\mathscr{S}$. Then X' and X^* are the same, and π still is not one to one; but $S^*=\{p, q, \infty\} \neq \{p, q\}=S^-$ so $\mathscr{S}^-$ is not subbase regular.

Example 5. Let X be the real line and let $\mathscr{S}$ consist of all closed connected subsets. Then $X'=X^*$ is the two point compactification. Finite intersections are screened from members. So $\mathscr{S}^-$ and $\mathscr{S}'$ are subbase regular and π is one to one. $\mathscr{S}$ is basic, and X^* is locally connected.

Example 6. Let X be the plane. Trivially, (or because of [6]) the collection of all closed connected subsets is a GA subbase $\mathscr{S}$. $\mathscr{S}$ is basic by theorems in [6]. Let S be the complement of the union of small open discs around integers on the S-axis. Let $T_{1,m}$ $m=1, 2, \ldots$, consist of the line $y=1$ and the segments $\{(n, y): 0 \leqslant y \leqslant 1\}$, $n=m, m+1, \ldots$. Let $T_{2,n}$ be the reflection of $T_{1,m}$ through the X-axis. S cannot be screened from $T_{1,m} \cup T_{2,m}$, but $S+$ ($=S$ intersected with the upper half-plane), and $S-$ ($=S$ intersected with the lower half-plane) can be. If ξ' is any maximal centered system containing all $T_{i,m}$, $i=1, 2$; $m=1, 2, \ldots$, then $\xi^* \in S^*-S^-$ so $\mathscr{S}^-$ is not subbase regular. Because of theorems in [7] neither X' nor X^* is locally connected at ξ^*.

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